Once we're at a triangle we have

$$\chi(\Delta) = V - E + F = 3 - 3 + 1 = 1$$

But, remember, we deleted a face at the beginning, so

$$\chi(P) = \chi(\Delta) + \overset{\text{the deleted face}}{\overset{\downarrow}{1}} = 2$$

$\square$

We say that a polyhedron is <u>regular</u> if all faces are congruent regular polygons. A polyhedron is a <u>Platonic solid</u> if it is convex and regular, and the same number of faces meet at every vertex.

<u>Theorem</u>: There are exactly 5 Platonic solids.

<u>proof</u>: Let P be a Platonic solid. Then, Euler's polyhedron formula tells us

$$\chi(P) = V - E + F = 2$$

Suppose that the faces of P have n sides and that c faces meet at every vertex.

Then $n \geq 3$ since a regular polygon has at least 3 sides, and $c \geq 3$ since it doesn't make sense for only 2 (or 1) faces to meet at a vertex.

We can also find relationships between V, E, and F.

$$nF = \#\left\{\left(\text{faces}, \begin{matrix}\text{edge of}\\ \text{face}\end{matrix}\right)\right\} = 2E$$

$$\Rightarrow F = \frac{2}{n}E$$

$$2E = \#\left\{\left(\text{edge}, \begin{matrix}\text{vertex}\\ \text{of edge}\end{matrix}\right)\right\} = cV$$

$$\Rightarrow V = \frac{2}{c}E$$

$$\Rightarrow V - E + F = \frac{2}{c}E - E + \frac{2}{n}E = E\left(\frac{2}{c} - 1 + \frac{2}{n}\right) = 2$$

Since $E > 0$,

$$\frac{2}{c} - 1 + \frac{2}{n} > 0$$

Now, since $c \geq 3$,

$$0 < \frac{2}{c} - 1 + \frac{2}{n} \leq \frac{2}{3} - 1 + \frac{2}{n} = \frac{2}{n} - \frac{1}{3}$$

$$\Rightarrow \frac{2}{n} > \frac{1}{3} \Rightarrow 6 > n \Rightarrow n = 3, 4, 5$$

$\underline{n=3}$: $\frac{2}{c} - 1 + \frac{2}{3} > 0 \Rightarrow \frac{2}{c} - \frac{1}{3} > 0 \Rightarrow \frac{2}{c} > \frac{1}{3}$

$\Rightarrow c < 6 \Rightarrow c = 3, 4, 5$

$\underline{n=4}$: $\frac{2}{c} - 1 + \frac{1}{2} = \frac{2}{c} - \frac{1}{2} > 0 \Rightarrow \frac{2}{c} > \frac{1}{2}$

$\Rightarrow 4 > c \Rightarrow c = 3$

$\underline{n=5}$: $\frac{2}{c} - 1 + \frac{2}{5} = \frac{2}{c} - \frac{3}{5} > 0 \Rightarrow \frac{2}{c} > \frac{3}{5}$

$\Rightarrow 10 > 3c \Rightarrow \frac{10}{3} > c \Rightarrow c = 3$

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n	c	Solid
3	3	Tetrahedron
3	4	Octahedron
3	5	Icosahedron
4	3	Cube
5	3	Dodecahedron

$\square$